3 Exponential and Logarithmic Functions

3.4 Solving Exponential and Logarithmic Equations

What You Should Learn

- Solve simple exponential and logarithmic equations.
- Solve more complicated exponential equations.
- Solve more complicated logarithmic equations.
- Use exponential and logarithmic equations to model and solve real-life problems.

Introduction

Introduction

Just read this slide and the next two. You do not need to write them down.

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties.

For $a > 0$ and $a \neq 1$ the following properties are true for all x and y for which

$\log_a x$ *and* $\log_a y$

are defined.

Introduction

One-to-One Properties

$a^x = a^y$ if and only if $x = y$.

$\log_a x = \log_a y$ if and only if $x = y$.

Inverse Properties

$$a^{\log_a x} = x$$

$$\log_a a^x = x$$

Example 1 – *Solving Simple Exponential and Logarithmic Exponential*

	Original Equation	*Rewritten Equation*	*Solution*	*Property*
a.	$2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b.	$\log_4 x - \log_4 8 = 0$	$\log_4 x = \log_4 8$	$x = 8$	One-to-One
c.	$\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
d.	$\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
e.	$e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
f.	$\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
g.	$\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse
h.	$\log_3 x = 4$	$3^{\log_3 x} = 3^4$	$x = 81$	Inverse

Introduction

The strategies used in Example 1 are summarized as follows.

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Solving Logarithmic Equations

Solving Logarithmic Equations

To solve a logarithmic equation, you can write it in exponential form.

$\ln x = 3$	Logarithmic form
$e^{\ln x} = e^3$	Exponentiate each side.
$x = e^3$	Exponential form

This procedure is called *exponentiating* each side of an equation. It is applied after the logarithmic expression has been isolated. Another option would be to convert from logarithmic form to exponential form using the “swoop”.

Example 6 – *Solving Logarithmic Equations*

Solve each logarithmic equation.

a. $\ln 3x = 2$

b. $\log_3(5x - 1) = \log_3(x + 7)$

Solution:

a. $\ln 3x = 2$ Write original equation.

$e^{\ln 3x} = e^2$ Exponentiate each side.

Example 6 – *Solution*

cont'd

$3x = e^2$ Inverse Property

$x = \frac{1}{3}e^2$ Multiply each side by $\frac{1}{3}$.

$x \approx 2.46$ Use a calculator.

The solution is $x = \frac{1}{3}e^2$

≈ 2.46

Check this in the original equation.

Example 6 – *Solution*

cont'd

b. $\log_3(5x - 1) = \log_3(x + 7)$ Write original equation.

$5x - 1 = x + 7$ One-to-One Property

$x = 2$ Solve for x

The solution $x = 2$. Check this in the original equation.

Solving Logarithmic Equations

Only write down the red part on this slide:

Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations.

The reason the domain does not include all real numbers is because you cannot take the log of a negative number. We will discuss this further in class.

Example 10 – *The Change of-Base Formula*

Prove the change-of-base formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$.

Read this slide and the next, but do not write them down.

Solution:

Begin by letting

$$y = \log_a x$$

and writing the equivalent exponential form

$$a^y = x.$$

Example 10 – *Solution*

cont'd

Now, taking the logarithms *with base b* of each side produces the following.

$$\log_b a^y = \log_b x$$

$$y \log_b a = \log_b x$$ Power Property

$$y = \frac{\log_b x}{\log_b a}$$ Divide each side by $\log_b a$.

$$\log_a x = \frac{\log_b x}{\log_b a}$$ Replace with $\log_a x$.

Applications

Example 12 – *Doubling an Investment*

You have deposited \$500 in an account that pays 6.75% interest, compounded continuously. How long will it take your money to double?

Solution:

Using the formula for continuous compounding, you can find that the balance in the account is

$$A = Pe^{rt}$$

$$= 500e^{0.0675t}.$$

To find the time required for the balance to double, let A = 1000, and solve the resulting equation for t.

Example 12 – *Solution*

cont'd

$500e^{0.0675t} = 1000$ Substitute 1000 for *A*.

$e^{0.0675t} = 2$ Divide each side by 500.

$\ln e^{0.0675t} = \ln 2$ Take natural log of each side.

$0.0675t = \ln 2$ Inverse Property

$$t = \frac{\ln 2}{0.0675}$$ Divide each side by 0.0675.

$t \approx 10.27$ Use a calculator.

The balance in the account will double after approximately 10.27 years.